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Rayleigh Scattering and the Impulse Response of Optical Fibers

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The impulse response of multimode optical fibers is distorted because each mode carries the signal at a different group velocity. Mode coupling tends to reduce the width of the impulse response. Rayleigh scattering, being the most fundamental scattering process in optical fibers, serves as a mode-coupling mechanism. However, it also causes radiation loss. The penalty of a seemingly apparent improvement of the impulse response through Rayleigh scattering is calculated in this paper. We conclude that, because of the high loss penalty, Rayleigh scattering is not a suitable technique for pulse-width improvement.

I. INTRODUCTION

The term "Rayleigh scattering" describes light scattering from refractive index inhomogeneities whose linear dimensions are much shorter than the wavelength of light. Most of the scattered light escapes from the core region of the fiber and enters the cladding or the space outside of the fiber. Some of the scattered power goes into other guided modes. Rayleigh scattering thus contributes to the losses in the fiber and also influences the impulse response through mode coupling.

Since mode coupling tends to improve the impulse response of optical fibers,[1,2] the question may be asked: How beneficial is Rayleigh scattering for light transmission in multimode fibers because of its mode-coupling ability? To answer this question we investigate the loss penalty that is incurred if Rayleigh scattering is assumed as the only mode-coupling mechanism.

For simplicity, our study is limited to a slab waveguide model (see Fig. 1) assuming that there is no variation of the refractive index or the light field in the y direction. Ignoring coupling between guided

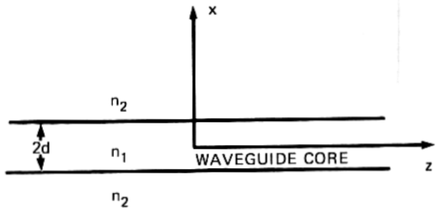

Fig. 1—Schematic of slab waveguide. The scattering centers are distributed randomly throughout the core and the outside medium. They are infinitely thin threads of slightly different refractive index extending in the y direction.

modes traveling in opposite directions, we calculate the width of the impulse response and the amount of scattering losses. These calculations allow us to establish the loss penalty. We find that the loss penalty for any significant pulse-width reduction caused by Rayleigh scattering is intolerably high. Thus, it is not feasible to improve the pulse dispersion of multimode fibers by intentionally implanting Rayleigh scatterers into the dielectric material of the fiber. However, improved pulse transmission is obtainable by using other carefully engineered mode-coupling mechanisms.[2]

II. THE COUPLING COEFFICIENT

The even guided TE modes of a slab waveguide consisting of a perfect dielectric are determined by the y component of its electric field.[3]

$$E_y = A \cos \kappa x \qquad |x| < d. \tag{1}$$

$$E_y = A \cos \kappa d\, e^{-\gamma(|x|-d)} \qquad |x| > d. \tag{2}$$

The odd guided modes are given by

$$E_y = A \sin \kappa d \qquad |x| < d. \tag{3}$$

$$E_y = \frac{x}{|x|} A \sin \kappa d\, e^{-\gamma(|x|-d)} \qquad |x| > d. \tag{4}$$

The magnetic field components are obtained by differentiation:

$$H_x = \frac{-i}{\omega\mu_0} \frac{\partial E_y}{\partial z} \tag{5}$$

and

$$H_y = \frac{i}{\omega\mu_0} \frac{\partial E_y}{\partial x}. \tag{6}$$

The factor $\exp [i(\omega t - \beta z)]$ is omitted from these and all subsequent field equations. The width of the core of the slab is $2d$. The parameters κ and γ are defined as follows:

$$\kappa = (n_1^2 k^2 - \beta^2)^{\frac{1}{2}} \tag{7}$$

and

$$\gamma = (\beta^2 - n_2^2 k^2)^{\frac{1}{2}}, \tag{8}$$

with $k = \omega\sqrt{\epsilon_0 \mu_0}$, n_1 = core index, and n_2 = cladding index. The propagation constant β is obtained as a solution of the eigenvalue equations:

$$\tan \kappa d = \frac{\gamma}{\kappa} \qquad \text{for even modes} \tag{9}$$

and

$$\tan \kappa d = -\frac{\kappa}{\gamma} \qquad \text{for odd modes.} \tag{10}$$

The amplitude coefficient is related to the power P carried by the mode

$$A = \left(\frac{2\gamma\omega\mu_0 P}{(1 + \gamma d)\beta} \right)^{\frac{1}{2}}. \tag{11}$$

In addition to guided modes, the slab with infinite cladding has radiation modes. The magnetic fields of the radiation modes follow from E_y by means of (5) and (6). The E_y component of the even radiation modes is[4]

$$E_y = B \cos \sigma x \qquad |x| < d \tag{12}$$

and

$$E_y = \left(\frac{2\omega\mu_0 P}{\pi\beta} \right)^{\frac{1}{2}} \cos \left[\rho(|x| - d) + \psi\right] \qquad |x| > d, \tag{13}$$

with ψ defined by

$$\tan \psi = \frac{\sigma \sin \sigma d}{\rho \cos \sigma d}. \tag{14}$$

The amplitude coefficient B is given by

$$B = \left(\frac{2\rho^2 \omega\mu_0 P}{\pi\beta(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)} \right)^{\frac{1}{2}}. \tag{15}$$

The parameters σ and ρ are defined by

$$\sigma = (n_1^2 k^2 - \beta^2)^{\frac{1}{2}} \tag{16}$$

and

$$\rho = (n_2^2 k^2 - \beta^2)^{\frac{1}{2}}. \tag{17}$$

Similarly, for the odd radiation modes we have

$$E_y = C \sin \sigma x \qquad |x| < d \tag{18}$$

and

$$E_y = \frac{x}{|x|}\left(\frac{2\omega\mu_0 P}{\pi\beta}\right)^{\frac{1}{2}} \sin\left[\rho(|x| - d) + \phi\right] \qquad |x| > d. \tag{19}$$

Phase ϕ is defined by

$$\tan\phi = \frac{\rho \sin \sigma d}{\sigma \cos \sigma d} \tag{20}$$

and the amplitude factor is given as

$$C = \left(\frac{2\rho^2\omega\mu_0 P}{\pi\beta(\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d)}\right)^{\frac{1}{2}}. \tag{21}$$

The coupling coefficient between two modes has the form[5–7]

$$K_{\nu\mu} = \frac{\omega\epsilon_0}{4iP}\int_{-\infty}^{\infty}(n^2 - n_0^2)E_\nu E_\mu^* dx. \tag{22}$$

E_ν and E_μ are the y components of the electric fields of two modes labeled ν and μ. The index distribution $n = n(x, z)$ describes the waveguide with slight random fluctuations around the average value, and $n_0 = n_0(x)$ is the index distribution that defines the ideal slab waveguide. It is $n_0 = n_1$ in the core and $n_0 = n_2$ outside. The ensemble average of $n^2 - n_0^2$ vanishes,

$$\langle n^2 - n_0^2\rangle = 0. \tag{23}$$

The power-coupling coefficients are obtained from the expression[7,8]

$$\begin{aligned} h_{\nu\mu} &= \frac{1}{L}\int_0^L dz\int_0^L dz' \langle K_{\nu\mu}(z)K_{\nu\mu}^*(z')\rangle e^{i(\beta_\mu-\beta_\nu)(z-z')} \\ &= \frac{\omega^2\epsilon_0^2}{16LP^2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\int_0^L dz\int_0^L dz' \langle(n^2 - n_0^2)(n'^2 - n_0'^2)\rangle \\ &\qquad \times E_\nu E_\nu'^* E_\mu' E_\mu^* e^{i(\beta_\mu-\beta_\nu)(z-z')}. \end{aligned} \tag{24}$$

The prime indicates quantities depending on x' and z'.

The purpose of this calculation is to study Rayleigh scattering. For this reason we may assume that the correlation of the index fluctuations reaches only over distances that are much smaller than the wavelength $2\pi/\beta_\nu$. The following correlation function is used:

$$\langle(n^2 - n_0^2)(n'^2 - n_0'^2)\rangle = D^2\langle(n^2 - n_0^2)^2\rangle\delta(x - x')\delta(z - z'), \tag{25}$$

where D is the correlation length of the index fluctuations. Substitution of (25) into (24) leads to

$$h_{\nu\mu} = \frac{\omega^2\epsilon_0^2}{16P^2}D^2\langle(n^2 - n_0^2)^2\rangle\int_{-\infty}^{\infty}|E_\nu|^2|E_\mu|^2 dx. \tag{26}$$

The remaining z integration (after integration over the delta function) over the distance L resulted in a factor L that canceled from the equation.

To evaluate the remaining integral in (26) we make the following assumption. All modes are considered sufficiently far from cutoff so that the guided mode fields are very weak at the core boundary, $x = d$, and negligible outside of the core. For a guide supporting very many modes, this assumption is justified for most of them. Thus, the integral in (26) effectively extends only over the region of the core. The integrals are of three different types:

$$I_1 = \int_{-d}^{d} \cos^2 \kappa_\nu x \cos^2 \kappa_\mu x \, dx, \tag{27}$$

$$I_2 = \int_{-d}^{d} \sin^2 \kappa_\nu x \sin^2 \kappa_\mu x \, dx, \tag{28}$$

and

$$I_3 = \int_{-d}^{d} \cos^2 \kappa_\nu x \sin^2 \kappa_\mu x \, dx. \tag{29}$$

Since almost all modes have rapidly oscillating fields in x direction inside of the core, we approximate these integrals by

$$I_1 = I_2 = I_3 \approx \frac{d}{2}. \tag{30}$$

With the help of (11) and (30) we obtain from (26)

$$h_{\nu\mu} = \frac{k^4 \gamma_\nu \gamma_\mu d}{8(1 + \gamma_\nu d)(1 + \gamma_\mu d)\beta_\nu \beta_\mu} D^2 \langle (n^2 - n_0^2)^2 \rangle. \tag{31}$$

In the spirit of our approximation, we may assume $\gamma_\nu d \gg 1$ and $\beta_\nu \approx n_1 k$, where n_1 indicates the core index. Thus, the power-coupling coefficient can be approximated as follows:

$$h = h_{\nu\mu} = \frac{k^2}{8 n_1^2 d} D^2 \langle (n^2 - n_0^2)^2 \rangle. \tag{32}$$

In this far-from-cutoff approximation, the power-coupling coefficient is independent of mode number. Rayleigh scattering couples with equal strength all of the modes.

With the same type of approximations, we obtain from (11), (15), (26), and (30) the coupling coefficient between a guided mode labeled ν and an even radiation mode with propagation constant β:

$$h_\nu^{(e)}(\beta) = \frac{\rho^2 k^3 D^2 \langle (n^2 - n_0^2)^2 \rangle}{8 n_1 \pi |\beta| (\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)}. \tag{33}$$

Coupling to odd radiation modes leads to the same type of coupling coefficient, $h_\nu^{(o)}(\beta)$, except that $\cos \sigma d$ and $\sin \sigma d$ are now interchanged.

The power (scattering) loss coefficient for mode ν is

$$\alpha_\nu = 2\int_0^{n_2 k} [h_\nu^{(e)}(\beta) + h_\nu^{(o)}(\beta)] d\rho. \tag{34}$$

This expression can be justified as follows. The power-coupling coefficient indicates the amount of power flowing per unit length from the guided mode to each individual radiation mode. The sum of the contributions to all radiation modes gives the total loss. Since radiation modes form a continuum, the sum becomes an integral. The factor 2 in front of the integral indicates the doubling of the loss caused by power flowing not only into forward but also into backward traveling radiation modes. The integral over ρ can be converted to integration over β as follows:

$$\alpha_\nu = 2\int_0^{n_2 k} [h_\nu^{(e)}(\beta) + h_\nu^{(o)}(\beta)] \frac{\beta}{\rho} d\beta. \tag{35}$$

The integration includes only propagating radiation modes. The contribution of even and odd modes is very nearly the same, so that we use only the coupling coefficient (33) and double the factor in front of the integral:

$$\alpha_\nu = \frac{k^3 D^2 \langle (n^2 - n_0^2)^2 \rangle}{2\pi n_1} \int_0^{n_2 k} \frac{\rho d\beta}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d}. \tag{36}$$

To the approximation used in this analysis, the power-radiation-loss coefficient of the guided modes is independent of mode number.

An exact solution of the integral in (36) is hard to obtain. If we consider the fact that for large values of d the sine and cosine functions pass through many periods throughout the range of integration, we can replace the integrand by its average value over a few periods of the periodic functions. This average is

$$\left(\frac{\rho}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} \right)_{\text{average}} = \frac{1}{\sigma}. \tag{37}$$

It now remains to solve the integral:

$$\int_0^{n_2 k} \frac{d\beta}{\sigma} = \int_0^{n_2 k} \frac{d\beta}{\sqrt{n_1^2 k^2 - \beta^2}} = \arcsin \frac{n_2}{n_1}, \tag{38}$$

In most cases of practical interest the ratio n_2/n_1 is very close to unity so that we can approximate the integral by $\pi/2$. We thus obtain the following equation for the radiation power loss coefficient

$$\alpha = \alpha_\nu = \frac{k^3 D^2 \langle (n^2 - n_0^2)^2 \rangle}{4n_1} = 2n_1 khd. \tag{39}$$

The last part of the equation follows from (32).

III. CALCULATION OF IMPULSE RESPONSE

Pulse propagation in optical fibers can be described by the following equation for the average power[2,7]

$$\frac{\partial P_\nu}{\partial z} + \frac{1}{v_\nu}\frac{\partial P_\nu}{\partial t} = -\alpha_\nu P + \sum_{\mu=1}^{N} h_{\nu\mu}(P_\mu - P_\nu). \tag{40}$$

This system of coupled power equations holds only for modes traveling in the same direction. Rayleigh scattering scatters power in forward as well as backward directions; however, we must ignore the backward scattered power flowing into guided modes. Physically, it appears that this approximation should pose no difficulty, since only those modes that travel in near synchronism have a chance to interact thoroughly. Backward scattered power travels away from the pulse that created it; thus, it cannot alter the shape of the impulse response except, perhaps, by repeated reflections. Backward scattered power contributes mainly to the scattering losses. We have taken backward scattering into radiation modes into account, but the additional loss caused by backward scattering into guided modes contributes far less loss and is ignored in our treatment. Thus, we recognize that the approximation may lead to a slight underestimation of the total scattering loss.

To solve (40) we use the trial solution

$$P_\nu = A_\nu e^{-\sigma z} e^{i\omega[t-(n_1 z/c)]}. \tag{41}$$

Substitution into (40) leads to

$$A_\nu = h\left(\sum_{\mu=1}^{N} A_\mu\right) \Big/ \left[\alpha - \sigma + Nh + i\omega\left(\frac{1}{v_\nu} - \frac{n_1}{c}\right)\right]. \tag{42}$$

We used the fact that the loss coefficients and the coupling coefficients are independent of the mode number. The quantity N is the total number of modes.

We obtain the group velocity of the modes from an approximation of the propagation constant. Using[9]

$$\kappa_\nu d \approx \nu \frac{\pi}{2}, \tag{43}$$

we obtain from (7)

$$\beta \approx \left[n_1^2 k^2 - \left(\nu \frac{\pi}{2d} \right)^2 \right]^{\frac{1}{2}}. \tag{44}$$

The inverse group velocity of mode ν is

$$\frac{1}{v_\nu} = \frac{d\beta}{d\omega} = \frac{1}{c} \frac{d\beta}{dk} = \frac{n_1^2 k}{c\beta}. \tag{45}$$

Using $n_1 k \gg \nu\pi/2d$ we obtain approximately

$$\frac{1}{v_\nu} \approx \frac{n_1}{c} (1 + G\nu^2), \tag{46}$$

with

$$G = \frac{\pi^2}{8(n_1 kd)^2}. \tag{47}$$

The solution of the equation system (42) is accomplished with ease, since we realize that the sum term in the numerator is independent of the mode label. Thus, the coefficients A_ν must be of the form

$$A_\nu = \frac{C}{\alpha - \sigma + Nh + i\omega \frac{n_1}{c} G\nu^2}. \tag{48}$$

Substitution of (48) into (42) leads to an eigenvalue equation for the determination of σ:

$$h \sum_{\mu=1}^{N} \frac{1}{\alpha - \sigma + Nh + i\omega \frac{n_1}{c} G\mu^2} = 1. \tag{49}$$

The sum can be approximated by the integral

$$\int_0^N \frac{dx}{\alpha - \sigma + Nh + i\omega \frac{n_1}{c} Gx^2} = \frac{1}{\left[i\omega \frac{n_1}{c} G(\alpha - \sigma + Nh) \right]^{\frac{1}{2}}} \times \arctan \left(\frac{i\omega \frac{n_1}{c} GN^2}{\alpha - \sigma + Nh} \right)^{\frac{1}{2}}. \tag{50}$$

Thus, we obtain from (49) and (50) the eigenvalue equation

$$\left(\frac{i\omega \frac{n_1}{c} GN^2}{\alpha - \sigma + Nh} \right)^{\frac{1}{2}} = \tan \left\{ \frac{1}{h} \left(i\omega \frac{n_1}{c} G(\alpha - \sigma + Nh) \right)^{\frac{1}{2}} \right\}. \tag{51}$$

Fortunately, we need only the lowest-order eigenvalue since it has the significance of the steady-state loss of the system of coupled modes and also determines the shape of the impulse-response function.[2,7] The solution of (51) is accomplished by using the fact that the lowest-order eigenvalue must be close to the loss coefficient α. Thus, we set

$$\sigma = \alpha + \eta. \tag{52}$$

Next, we expand the tangent function in series and solve for η. In this way we obtain the approximate solution

$$\sigma = \alpha + \frac{4}{45}\left(\frac{n_1}{c}G\right)^2 \frac{N^3}{h}\,\omega^2 + i\omega\,\frac{n}{3c}\,GN^2. \tag{53}$$

For our purposes the coefficient ρ of ω^2 is of most importance. We obtain the general pulse shape by substituting (53) into (41) and integrating over ω from $-\infty$ to ∞. Neglecting the ω dependence of A_ν, we find a Gaussian-shaped pulse whose width is[2,7]

$$\Delta t = 4\sqrt{\rho L} = \frac{8}{\sqrt{45}}\,\frac{n_1}{c}\,G\,\frac{N^{3/2}}{\sqrt{h}}\,\sqrt{L}. \tag{54}$$

The width of the signal in the absence of mode coupling is

$$\Delta T = \left(\frac{1}{v_N} - \frac{1}{v_1}\right)L = \frac{n_1}{c}\,GN^2L. \tag{55}$$

The relative improvement of the width of the steady-state pulse in the presence of mode coupling is expressed by the factor[2,7]

$$R = \frac{\Delta t}{\Delta T} = \frac{8}{\sqrt{45NhL}}\cdot \tag{56}$$

Using (39) and (56), we define the loss penalty by the expression[2,7]

$$R^2\alpha L = 2.8\,\frac{n_1kd}{N}\cdot \tag{57}$$

The number of modes is obtained from (44) with the help of the cutoff condition $\beta = n_2k$ for $\nu = N$; thus,

$$N = \frac{2kd}{\pi}\sqrt{n_1^2 - n_2^2}. \tag{58}$$

The expression for the loss penalty thus assumes the form

$$R^2\alpha L = \frac{4.4}{\sqrt{1 - \dfrac{n_2^2}{n_1^2}}} \approx \frac{3.1}{\sqrt{1 - \dfrac{n_2}{n_1}}}\cdot \tag{59}$$

IV. DISCUSSION

We can now answer the question that was asked in the introduction: Is Rayleigh scattering significantly beneficial because of its ability to shorten the width of the impulse response? Let us assume that we have a slab waveguide with a core-to-cladding index ratio of $n_1/n_2 = 1.01$. From (59) we obtain in this case

$$R^2 \alpha L = 31.2 = 135 \text{ dB}. \tag{60}$$

We may now ask how much loss is associated with a relative decrease of the width of the impulse response by a factor 2, or $R = 0.5$. We see from (60) that the amount of scattering loss associated with this "improvement" is

$$\alpha L = 540 \text{ dB}. \tag{61}$$

This shows that if we are hoping for a reduction in the width of the impulse response with the help of Rayleigh scattering, we have to pay an intolerably high price in added loss. Since Rayleigh scattering losses are known to be quite small, (59) indicates that this mechanism does not help to reduce the width of the impulse response under ordinary conditions.

We are thus forced to consider Rayleigh scattering as detrimental to light transmission in optical fibers. Fortunately, it is a small effect that does not provide prohibitively high losses at visible or infrared wavelength.

It is easy to understand why Rayleigh scattering is not more effective in reducing the width of the impulse response. It has been shown that a very carefully shaped power spectrum of the function describing fiber irregularities is required to reduce the loss penalty for pulse width reduction.[2] Rayleigh scattering is particularly poorly suited for this purpose since its power spectrum is flat. Only a very small fraction of the total amount of scattering is used for mode mixing, most of it is used for light scattering into radiation modes leading to scattering losses.

Our calculation was based on a slab waveguide model. However, the result is expected to be representative of round optical fibers. Experience has shown that estimates of the performance of round fibers can be obtained from scattering data calculated on the basis of a slab waveguide model.

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REFERENCES

1. S. D. Personick, "Time Dispersion in Dielectric Waveguides," B.S.T.J., *50*, No. 3 (March 1971), pp. 843–859.
2. D. Marcuse, "Pulse Propagation in Multimode Dielectric Waveguides," B.S.T.J., *51*, No. 6 (July-August 1972), pp. 1199–1232.
3. D. Marcuse, *Light Transmission Optics*, New York: Van Nostrand-Reinhold Co., 1972, pp. 307–309.
4. Ref. 3, pp. 314–318. The notation for the field in the cladding is written in more compact form in this paper.
5. A. W. Snyder, "Coupled Mode Theory for Optical Fibers," J. Opt. Soc. Am., *62*, No. 11 (November 1972), pp. 1267–1277.
6. D. Marcuse, "Coupled Mode Theory of Round Optical Fibers," B.S.T.J., *52*, No. 6 (July-August 1973), pp. 817–842.
7. D. Marcuse, *Theory of Dielectric Optical Waveguides*, New York: Academic Press, 1974.
8. D. Marcuse, "Derivation of Coupled Power Equations," B.S.T.J., *51*, No. 1 (January 1972), pp. 229–237.
9. Ref. 3, p. 328.